Time-delay-induced stabilization of coupled discrete-time systems

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This paper shows that the time-delay-induced stabilization occurs in discrete-time systems on numerical simulations. The stability analysis proves that this phenomenon never occurs in the discrete-time systems that have an odd-number property. This property is well known as the weak point of the delayed feedback control of chaos. Furthermore, we show that the phenomenon never occurs in any one-dimensional discrete-time system.

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Coupled nonlinear oscillators show several interesting phenomena both on numerical and real experiments. The phenomena have been investigated in a wide range of fields. The dynamics of weak-coupled oscillators can be described by the phase dynamics which is useful for theoretical analysis. In the case of strong coupling, however, we have to deal with not only the phase but also the amplitude of oscillation. In such case, it was reported that *amplitude death* can occur in coupled oscillators. This phenomenon is a coupling-induced stabilization of the origin in the oscillators [1,2]. For two coupled oscillators, Aronson, Ermentrout, and Kopell have theoretically investigated this phenomenon in detail [3].

In real coupled systems, there exists a time-delay effect due to the finite speed of data propagation. Nevertheless, most studies on coupled oscillators did not consider the time-delay coupling. In recent years, several researchers have studied the time-delay coupled oscillators [4–6]. Reddy, Sen, and Johnston [7] showed that the time-delay coupling induces the amplitude death even for two identical oscillators. This phenomenon can be considered as a time-delay-induced stabilization of the origin in the coupled systems. Their result has gained more and more attention [8,9]. The time-delay-induced amplitude death has been investigated in detail [10]; furthermore, experimental observations of electronic circuits [11], living oscillators [12], and the thermo-optical oscillators [13] have been reported.

In the present paper, we investigate a time-delay-induced stabilization of steady states in two identical discrete-time systems coupled by a delay connection. Our main purposes are as follows: (i) we observe the stabilization on numerical simulations; (ii) we prove that the stabilization without time delay never occurs; (iii) we prove that the stabilization never occurs in the class of discrete-time systems. The feature of this paper is that our results do not depend on the degree of systems; in other words, they can be valid not only for low- but also high-dimensional systems.

Let us consider two identical discrete-time subsystems $\Sigma_{\alpha,\beta}$,

$$\Sigma_\alpha : \begin{cases} \boldsymbol{x}_\alpha(n+1) = \boldsymbol{f}[\boldsymbol{x}_\alpha(n)] + \boldsymbol{b} u_\alpha(n) \\ y_\alpha(n) = \boldsymbol{g}[\boldsymbol{x}_\alpha(n)], \end{cases}$$

$$\Sigma_\beta : \begin{cases} \boldsymbol{x}_\beta(n+1) = \boldsymbol{f}[\boldsymbol{x}_\beta(n)] + \boldsymbol{b} u_\beta(n) \\ y_\beta(n) = \boldsymbol{g}[\boldsymbol{x}_\beta(n)], \end{cases}$$

where $\boldsymbol{x}_{\alpha,\beta}(n) \in \boldsymbol{R}^m$ are the system variables, $u_{\alpha,\beta}(n) \in \boldsymbol{R}$ and $y_{\alpha,\beta}(n) \in \boldsymbol{R}$ are the input and output signals. $\boldsymbol{f}: \boldsymbol{R}^m \to \boldsymbol{R}^m$ denotes the nonlinear function. $\boldsymbol{b} \in \boldsymbol{R}^m$ is the input matrix and $\boldsymbol{g}: \boldsymbol{R}^m \to \boldsymbol{R}$ is the output function. These subsystems $\Sigma_{\alpha,\beta}$ are coupled by

$$u_\alpha(n) = \varepsilon\{y_\beta(n-\tau) - y_\alpha(n)\}, \tag{1a}$$

$$u_\beta(n) = \varepsilon\{y_\alpha(n-\tau) - y_\beta(n)\}. \tag{1b}$$

$\varepsilon \in \boldsymbol{R}$ is coupling strength and $\tau > 0$ denotes the delay time (see Fig. 1). It should be noticed that each of input signals $u_{\alpha,\beta}(n)$ include other output delayed signals $y_{\beta,\alpha}(n-\tau)$. The steady state of subsystems $\Sigma_{\alpha,\beta}$ without coupling ($\varepsilon = 0$) is given by $\boldsymbol{x}_f = \boldsymbol{f}(\boldsymbol{x}_f)$. The location of steady state $\boldsymbol{x}_f$ never changes even by delayed coupling; in other words, the delayed coupling changes only the stability of state.

For the first example, we use the delayed logistic subsystems [14],

$$\Sigma_\alpha : \begin{cases} \begin{bmatrix} x_{\alpha 1}(n+1) \\ x_{\alpha 2}(n+1) \end{bmatrix} = \begin{bmatrix} x_{\alpha 2}(n) \\ p x_{\alpha 2}(n)\{1 - x_{\alpha 1}(n)\} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_\alpha(n) \\ y_\alpha(n) = x_{\alpha 1}(n), \end{cases} \tag{2a}$$

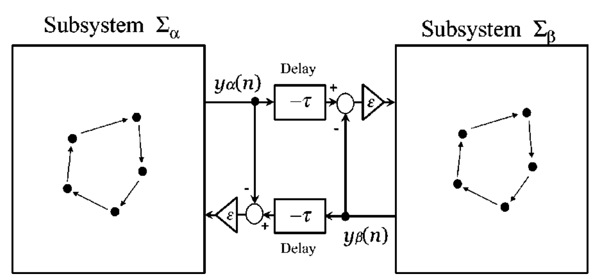

FIG. 1. Delay coupled discrete-time systems.

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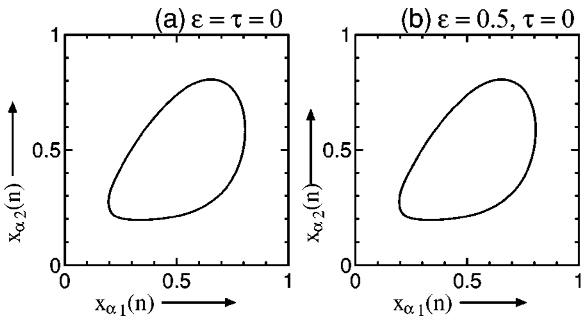

FIG. 2. Behavior of systems ($p=2.1$). (a) Isolated subsystem ($\varepsilon=0$) and (b) nondelay coupling ($\varepsilon=0.5,\ \tau=0$) system Σ_α.

$$\Sigma_\beta:\begin{cases}\begin{bmatrix}x_{\beta1}(n+1)\\ x_{\beta2}(n+1)\end{bmatrix}=\begin{bmatrix}x_{\beta2}(n)\\ px_{\beta2}(n)\{1-x_{\beta1}(n)\}\end{bmatrix}+\begin{bmatrix}1\\0\end{bmatrix}u_\beta(n)\\ y_\beta(n)=x_{\beta1}(n),\end{cases} \tag{2b}$$

coupled by (1). The steady state is $\boldsymbol{x}_f=[(p-1)/p,(p-1)/p]^T$. The parameter is fixed at $p=2.1$. Figure 2(a) shows the behavior of subsystem Σ_α without coupling ($\varepsilon=0$). The limit cycle (quasiperiodic orbit) is observed around the unstable steady state $\boldsymbol{x}_f$. This cycle is maintained for the nondelay coupling ($\tau=0,\ \varepsilon=0.5$) as shown in Fig. 2(b). The bifurcation diagram of the nondelay coupled systems for the range $\varepsilon\in[0,1]$ is shown in Fig. 3(a): it can be seen that there does not exist stabilization. On the other hand, the delay coupling induces the stabilization of the steady state for a range of ε [see Fig. 3(b)]. This is the time-delay-induced stabilization in discrete-time systems; therefore, we can accomplish the first purpose of this paper.

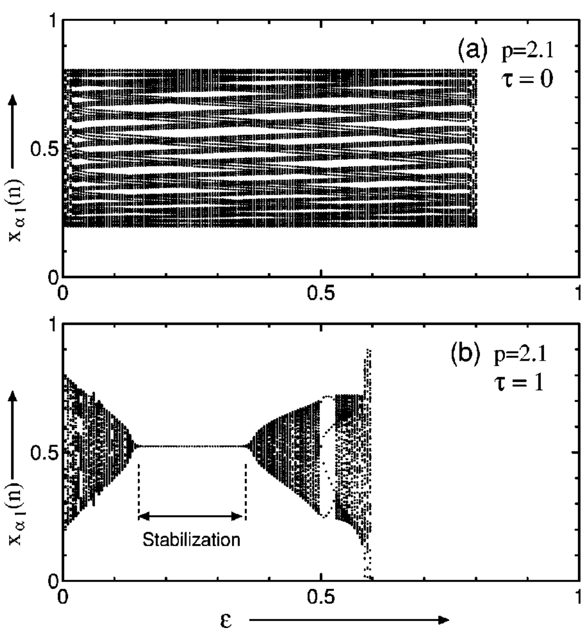

FIG. 3. Bifurcation diagram for ε. (a) Nondelay coupling ($\varepsilon>0,\ \tau=0$) and (b) delay coupling ($\varepsilon>0,\ \tau=1$).

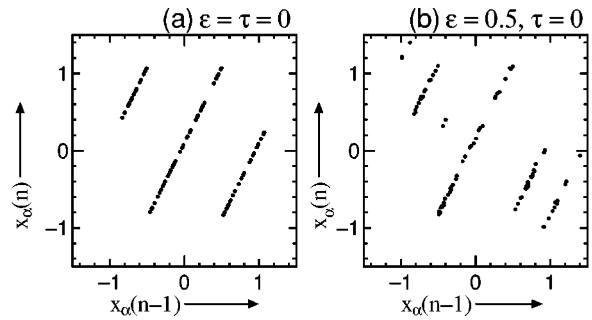

FIG. 4. Behavior of systems ($p_1=1.95,\ p_2=0.1$). (a) Isolated subsystem ($\varepsilon=0$) and (b) nondelay coupling ($\varepsilon=0.5,\ \tau=0$) system Σ_α.

For the second example, we use the following subsystems [15]:

$$\Sigma_\alpha:\begin{cases}x_\alpha(n+1)=f[x_\alpha(n)]+u_\alpha(n)\\ y_\alpha(n)=x_\alpha(n),\end{cases} \tag{3a}$$

$$\Sigma_\beta:\begin{cases}x_\beta(n+1)=f[x_\beta(n)]+u_\beta(n)\\ y_\beta(n)=x_\beta(n),\end{cases} \tag{3b}$$

coupled by (1). The nonlinear function is described by

$$f(x)=\begin{cases}p_1(x+1)+p_2, & x<-0.5\\ p_1x+p_2, & |x|\leq 0.5\\ p_1(x-1)+p_2, & x>0.5.\end{cases}$$

The steady state is $x_f=p_2/(1-p_1)$. Figure 4(a) indicates the chaotic behavior of isolated ($\varepsilon=0$) subsystem Σ_α for $p_1=1.95$, $p_2=0.1$. The behavior of the coupled system without delay ($\varepsilon=0.5,\tau=0$) is shown in Fig. 4(b). We plot the bifurcation diagram for the nondelay coupled system ($\tau=0$) in Fig. 5(a). Like the first example, the stabilization does not occur in a nondelay coupled system. Figures 5(b) and 5(c) show the bifurcation diagram of delayed coupled system for $\tau=1,2$. In contradiction to the first example, there does not exist a time-delay-induced stabilization in system (3).

We summarize the above numerical results: (1) the nondelay coupling does not cause stabilization for either system; (2) the time-delay-induced stabilization depends on the system structure. These results lead us to the following two problems. (a) Does the nondelay coupling not cause stabilization? (b) Under what condition does the time-delay-induced stabilization never occur? The problems (a) and (b) correspond to our main purposes (ii) and (iii). We shall solve these problems theoretically below.

Linearizing subsystems $\Sigma_{\alpha,\beta}$ around the steady state $\boldsymbol{x}_f$, we obtain

$$\Delta\Sigma_\alpha:\begin{cases}\boldsymbol{\xi}_\alpha(n+1)=\boldsymbol{A}\boldsymbol{\xi}_\alpha(n)+\boldsymbol{b}\Delta u_\alpha(n)\\ \Delta y_\alpha(n)=\boldsymbol{c}\boldsymbol{\xi}_\alpha(n),\end{cases} \tag{4}$$

$$\Delta\Sigma_\beta:\begin{cases}\boldsymbol{\xi}_\beta(n+1)=\boldsymbol{A}\boldsymbol{\xi}_\beta(n)+\boldsymbol{b}\Delta u_\beta(n)\\ \Delta y_\beta(n)=\boldsymbol{c}\boldsymbol{\xi}_\beta(n),\end{cases} \tag{5}$$

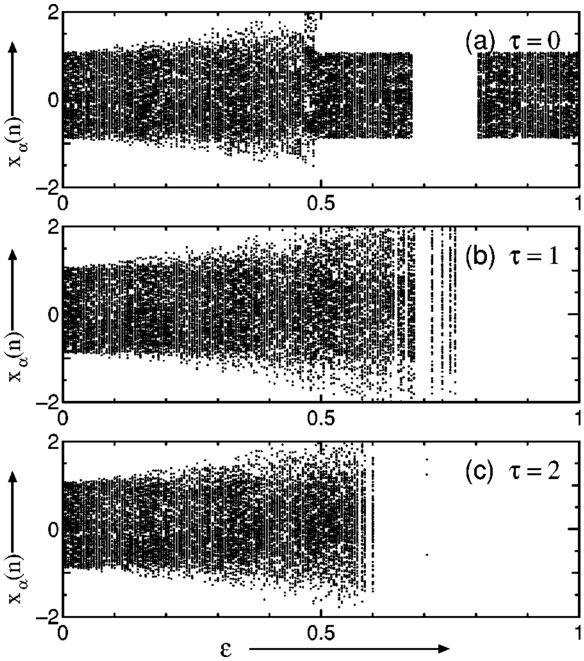

FIG. 5. Bifurcation diagram for ε. (a) Nondelay coupling ($\varepsilon>0,\ \tau=0$), (b) delay coupling ($\varepsilon>0,\ \tau=1$), and (c) delay coupling ($\varepsilon>0,\ \tau=2$).

coupled by

$$\Delta u_{\alpha}(n)=\varepsilon\{\Delta y_{\beta}(n-\tau)-\Delta y_{\alpha}(n)\}, \tag{6a}$$

$$\Delta u_{\beta}(n)=\varepsilon\{\Delta y_{\alpha}(n-\tau)-\Delta y_{\beta}(n)\}, \tag{6b}$$

where $\boldsymbol{\xi}_\alpha(n):=\boldsymbol{x}_\alpha(n)-\boldsymbol{x}_f$, $\boldsymbol{\xi}_\beta(n):=\boldsymbol{x}_\beta(n)-\boldsymbol{x}_f$, $\Delta y_\alpha(n):=y_\alpha(n)-\boldsymbol{g}(\boldsymbol{x}_f)$, $\Delta y_\beta(n):=y_\beta(n)-\boldsymbol{g}(\boldsymbol{x}_f)$, $\boldsymbol{A}:=d\boldsymbol{f}(\boldsymbol{x}_f)/d\boldsymbol{x}$, $\boldsymbol{c}:=d\boldsymbol{g}(\boldsymbol{x}_f)/d\boldsymbol{x}$. We assume that $(\boldsymbol{A},\boldsymbol{b},\boldsymbol{c})$ is controllable and observable. $\boldsymbol{A}$ is a Jacobi matrix at the steady state $\boldsymbol{x}_f$. The relation between the input $\Delta u_{\alpha,\beta}(n)$ and output $\Delta y_{\alpha,\beta}(n)$ signals of each system $\Delta\Sigma_{\alpha,\beta}$ can be described by frequency domain description:

$$Y_\alpha(z)=G(z)U_\alpha(z),\quad Y_\beta(z)=G(z)U_\beta(z).$$

$Y_{\alpha,\beta}(z),U_{\alpha,\beta}(z)$ are the z transformations of $\Delta y_{\alpha,\beta}(n)$, $\Delta u_{\alpha,\beta}(n)$ $[Y_{\alpha,\beta}(z):=\mathcal{Z}[\Delta y_{\alpha,\beta}(n)]$, $U_{\alpha,\beta}(z):=\mathcal{Z}[\Delta u_{\alpha,\beta}(n)]]$. The transfer function is given by

$$G(z)=\frac{n(z)}{d(z)}=\boldsymbol{c}(z\boldsymbol{I}_{m\times m}-\boldsymbol{A})^{-1}\boldsymbol{b},$$

where the characteristic polynomial is $d(z)=\det[z\boldsymbol{I}_{m\times m}-\boldsymbol{A}]$. It is well known that the eigenvalues of Jacobi matrix $\boldsymbol{A}$ are equivalent to the roots of $d(z)=0$. Hence, the stability of the steady state in isolated subsystems depends only on the characteristic polynomial $d(z)$.

The linearized dynamics of the coupled system consisting of $\Sigma_{\alpha,\beta}$ and coupling (1) can be described by the linear system consisting of $\Delta\Sigma_{\alpha,\beta}$ and coupling (6). The transfer function of the linear system is

$$H(z,\varepsilon,\tau)=\frac{\varepsilon z^{-\tau}G(z)\{1+\varepsilon G(z)\}}{\{1+\varepsilon G(z)-\varepsilon z^{-\tau}G(z)\}\{1+\varepsilon G(z)+\varepsilon z^{-\tau}G(z)\}}=\frac{\varepsilon z^{\tau}n(z)\{d(z)+\varepsilon n(z)\}}{h_1(z,\varepsilon,\tau)h_2(z,\varepsilon,\tau)},$$

where $h_1(z,\varepsilon,\tau)=z^\tau d(z)+\varepsilon n(z)(z^\tau-1)$, $h_2(z,\varepsilon,\tau)=z^\tau d(z)+\varepsilon n(z)(z^\tau+1)$. It should be noted that the stability of the steady state $\boldsymbol{x}_\alpha(n)=\boldsymbol{x}_\beta(n)=\boldsymbol{x}_f$ in the coupled system depends only on the characteristic polynomial of $H(z,\varepsilon,\tau)$ [i.e., $h_1(z,\varepsilon,\tau)h_2(z,\varepsilon,\tau)$].

In the case of nondelay coupling ($\tau=0,\ \varepsilon\neq0$), we have the transfer function

$$H(z,\varepsilon,0)=\frac{\varepsilon n(z)[d(z)+\varepsilon n(z)]}{d(z)^2}.$$

The characteristic polynomial of $H(z,\varepsilon,0)$ is $d(z)^2$. Therefore, if the steady state $\boldsymbol{x}_f$ in the isolated subsystems is unstable [i.e., $d(z)$ is an unstable polynomial], then the steady state $\boldsymbol{x}_\alpha(n)=\boldsymbol{x}_\beta(n)=\boldsymbol{x}_f$ in the nondelay coupled system is also unstable [i.e., $d(z)^2$ is an unstable polynomial]. As a result, we give an answer to the problem (a) mentioned above: if the steady state $\boldsymbol{x}_f$ in isolated subsystems is unstable, then the nondelay coupling never causes any systems' stabilization. A similar result was derived for the continuous time systems [3].

Now let us consider the second problem. The characteristic polynomial of $H(z,\varepsilon,\tau)$ can be described by $h_1(z,\varepsilon,\tau)h_2(z,\varepsilon,\tau)$, where $h_i(z,\varepsilon,\tau)$ $(i=1,2)$ are continuous in z. Furthermore, it is obvious that

$$\lim_{z\to+\infty}h_1(z,\varepsilon,\tau)=+\infty,\quad \forall\varepsilon\in[0,1],\quad \forall\tau\geqslant0. \tag{7}$$

For $z=1$, we have $h_1(1,\varepsilon,\tau)=d(1)$. If $d(z)=0$ has an odd number of real roots greater than 1, then $d(1)$ is less than zero. From Eq. (7) and $d(1)<0$, it is obvious that there exists at least one real root $z>1$ of $h_1(z,\varepsilon,\tau)=0$. Therefore, $h_1(z,\varepsilon,\tau)$ is an unstable characteristic polynomial for any (ε,τ). We summarize the above result as follows: If the Jacobi matrix $\boldsymbol{A}$ has an odd number of real eigenvalues greater than 1 (i.e., $\boldsymbol{A}$ has the odd-number property), then the stabilization never occurs for any $\tau>0$, $\varepsilon\neq0$.

We should note the following two points. First, this property is a sufficient condition for the steady state to be unstable. In other words, we cannot guarantee that the state is stable, even if the property is not satisfied. Second, this property is similar to the delayed feedback control (DFC) of chaos [16–19]. The derivation of our result has similarity to that in Ushio's paper [17].

First of all, we consider the delayed logistic system (2). The Jacobi matrix $\boldsymbol{A}$ around $\boldsymbol{x}_f$ is

$$\boldsymbol{A}=\begin{bmatrix}0 & 1\\ 1-p & 1\end{bmatrix}.$$

For $p=2.1$, we have

$$x_f=\begin{bmatrix}0.5238\\0.5238\end{bmatrix},\quad A=\begin{bmatrix}0 & 1\\-1.1 & 1\end{bmatrix}.$$

The steady state x_f is unstable because of the eigenvalues $\lambda=0.5\pm i\sqrt{3.4}/2$ outside of the unit circle. From the answer to problem (a), we can guarantee that the nondelay coupling never causes the stabilization in system (2). This theoretical result agrees with Fig. 3(a). In contrast, we cannot guarantee stabilization caused by the delay coupling, since A does not have the odd-number property.

Second, we consider system (3) as an example. For $p_1=1.95$ and $p_2=0.1$, the steady state and Jacobi matrix are $x_f=-0.1053$, $A=1.95$. The steady state x_f is unstable due to $\lambda=1.95>1$. From the answer to problem (a), we can guarantee that the nondelay coupling never causes the stabilization in system (3). This theoretical result agrees with Fig. 5(a). Furthermore, we can guarantee that the delay coupling never causes the stabilization in system (3), since A has the odd-number property. This result agrees with the numerical simulations in Figs. 5(b) and 5(c).

This paper has considered the sufficient conditions for steady state to be unstable; while we have not derived the necessary and sufficient condition yet. It would be possible for us to derive such condition by applying the Schur stability algorithm [20] to the characteristic polynomial $h_1(z,\varepsilon,\tau)h_2(z,\varepsilon,\tau)$; however, the derivation of the simple and general condition is not easy.

Now let us consider the problem of stability of the steady state in one-dimensional subsystems. The Jacobi matrix satisfies $|A|>1$, since the isolated subsystems ($\varepsilon=0$) behave oscillatory. For $A>1$, the odd-number property is satisfied, so that the polynomial $h_1(z,\varepsilon,\tau)h_2(z,\varepsilon,\tau)$ is unstable. For $A<-1$, we shall discuss below. In the case of even τ, we obtain

$$\lim_{z\to-\infty} h_1(z,\varepsilon,\tau)=-\infty.$$

If the polynomial $h_1(-1,\varepsilon,\tau)$ satisfies $h_1(-1,\ \varepsilon,\tau)=d(-1)=-1-A>0$, then $h_1(z,\varepsilon,\tau)=0$ has at least one root less than -1. In other words, for even τ and $A<-1$, the characteristic polynomial is unstable. On the other hand, in the case of odd τ, we have

$$\lim_{z\to-\infty} h_2(z,\varepsilon,\tau)=+\infty.$$

If the polynomial $h_2(-1,\varepsilon,\tau)$ satisfies $h_2(-1,\varepsilon,\tau)=-d(-1)=1+A<0$, then $h_2(z,\varepsilon,\tau)=0$ has at least one root less than -1. As a result, for odd τ and $A<-1$, the characteristic polynomial is unstable.

We summarize the above discussions: the time-delay-induced stabilization never occurs in any one-dimensional subsystem. This result agrees with the numerical example of system (3). From this theoretical result and the numerical example of system (2), we notice that at least two variables are required for the stabilization.

This paper has investigated the time-delay-induced stabilization of two identical coupled discrete-time systems, and has shown two theoretical results. On the basis of our results, we think that the mechanism of stabilization is related to suppression of chaos by the DFC. Hence, the results of recent research in DFC would be used for investigation of time-delay-induced stabilization. Furthermore, our approach in this paper may be useful for large size coupled systems.

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